Keynes-Ramsey's century-old dispute: a reinterpretation by lattice theory and a coherent solution

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Abstract

The subject of this paper is to set up the formal structure that puts in evidence in what the keynesian theory of probability does not conform to Ramsey's one. Following the seminal paper of Birkhoff and von Newmann about Quantum Mechanics (1936), this aim is obtained by identifying the algebraic-axiomatic properties that support the relation between belief and probability in Keynesian and Ramseyan theories. The paper shows that there is a particular class of abstract algebras able to represent Keynes's problem and shared properties with traditional Ramsey's probability theory. Such a class of abstract algebras is called a bounded distributive lattice. Introducing the notion of interval probability measure it is shown that, assuming a model of uncertainty, keynesian uncertain beliefs are represented by an isomorphic interval of probability measure. Such a result is a coherent response to the Ramseyian question.

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1 Introduction

John Maynard Keynes published *A Treatise on Probability* (*TP*) in 1921, mostly an elaboration of his 1901 dissertation. Keynes worked on *TP* several times: its almost definitive form was largely completed by 1911, while first typeset was ready in 1913 and it was published, after the end of WWI, in 1921. *TP* introduced a theory of probability alternative to the contemporary frequentist approach and suggested a theory of uncertainty, as a notion different from risk, that was reported in *The General Theory of Employment, Interest and Money* in 1936.

In 1922, Frank Ramsey, still a second year undergraduate in mathematics, reviewed *TP* arguing against Keynes's theory of probability. A few years later, he put forward his own theory of probability in *Truth and Probability* (1926). Ramsey introduced the subjective notion of probability based on the idea of probability as the willingness to bet and proposed an axiomatization of what can be considered the seminal formalization of Subjective Expected Utility (1926, 178-179), later developed by de Finetti and Savage.

Since the 1920s Keynes and Ramsey's theories of probability have been considered irreconcilable.[1] This paper submits that this judgment is misplaced.

The century-old dispute among scholars has often involved mathematical concepts such as measurability, rules of conduct, axioms, but it has never referred to Keynes and Ramsey's theoretical architecture that supported the representation of beliefs by probabilities.

In such a perspective, the paper starts from the premise about Keynes's notion of probability that Ramsey introduces in *Chapter 2* of *Truth and Probability*. Ramsey sets out that the degree of the probability relation is the same as the degree of belief and that both of them are expressed by numbers that are the same. "But if, as Mr. Keynes holds, these things are not always expressible by numbers, then we cannot give his statement that the degree of the one is the same as the degree of the other such a simple interpretation, but must suppose him to mean only that there is a one-to-one correspondence between probability relations and the degrees of belief which they justify. This correspondence must clearly preserve the relations of greater and less, and so make the manifold of probability relations and that of degrees of belief similar in Mr. Russell's sense" (1926, 160).[2]

Ramsey assumes that there is the one-to-one relation between degree of belief and probability.[3] In his framework there are two ordered sets: the former of precise beliefs and the latter of fully reliable and unique probabilities. In

[1] The 1920s was one of the most exciting period in the foundation of sciences: the intuitionist-formalist controversy between Brouwer and Hilbert, respectively, will influence the development of mathematics, in physics the Einstein-Bohr's controversy on the foundation of quantum mechanics will conditioning the following decades, in economics the controversy between Keynes and Ramsey will determine the theory of optimal decision.

[2] "I think it is a pity that Mr. Keynes did not see this clearly, because the exactitude of this correspondence would have provided quite as worthy material scepticism as did the numerical measurement of probability relations. Indeed some of his arguments against their numerical measurement appear to apply quite equally well against their exact correspondence with degrees of belief; for instance, he argues that if rates of insurance correspond to subjective, i.e. actual, degrees of belief, these are not rationally determined, and we cannot infer that probability relations can be similarly measured. It might be argued that the true conclusion in such a case was not that, as Mr. Keynes thinks, to the non-numerical probability relation corresponds a non-numerical degree of rational belief, but that degrees of belief, which were always numerical, did not correspond one to one with the probability relations justifying them. For it is, I suppose, conceivable that degrees of belief could be measured by a psychogalvanometer or some such instrument, and Mr. Keynes would hardly wish it to follow that probability relations could all be derivatively measured with the measures of the beliefs which they justify" (Ramsey 1926, 160).

[3] In a nutshell one-to-one correspondence between belief and numerical assignment (probability) implies that they never add or lose any information.

these sets, elements are compared pair-wise and the order relation is defined. As a consequence, the one-to-one relation between belief and probability is an order bijective function or isomorphism, that is the function clearly preserves the relations of greater and less among elements of the sets.[4]

Inexplicably, for a century, the seminal problem set out by Ramsey has been never considered. Nevertheless, it is the formal core of the Ramseyan doubt about the Keynesian notion of non-numerical probability.

Levi (2010) proposed a rational agent that is committed to a state of full belief belonging to a space of potential states of full belief that are relevant for the agent at that time. Following this intuition, the paper points out that it is possible to have a family of functions and given some conditions a unique one-to-one relationship between degrees of belief and degrees of probability, even if they are both imprecise and non-numerical, as Keynes sets out. From a theoretical point of view, the paper submits that the Keynesian framework, that supports the notion of non-numerical, imprecise or not fully reliable and then non necessarily additive probability, can be shown as a general representation of the Ramseyan framework that defines a unique and fully reliable, then additive, probability function.[5]

Keynes assumes that there are two sets: the former of imprecise beliefs and the latter of non-numerical probabilities, represented by intervals of real numbers or approximate probabilities (Keynes 1921, Ch15; Russell 1922).[6] Crucially, also in this case, it should exist the one-to-one relation between the set of vague belief and the set of real numbers or interval of probabilities: an isomorphism, as Ramsey requires.[7]

The object of this paper is to set up the formal structure that puts in evidence

[4] Given two ordered sets M and N are isomorphic or $M \cong N$ if there exists a map φ from M onto N such that:

for all x, y in M $x \leq y$ iff $\varphi(x) \leq \varphi(y)$ in N.

Then, φ is an *order-isomorphism* or *bijection* between two ordered sets. Such ordered one-to-one correspondence must preserve join and met whenever they exist.

[5] Levi (2010) sets out that "Ramsey (1926) contended that the principles of rational probability judgment constitute a logic of consistency without a logic of true, Ramsey explicitly insisted that probability logic is a logic of consistency...Although he sought also to articulate a logic of true for probability, he was quite clear, as was de Finetti and Savage, that creedal probability does not carry truth-value so that a logic of truth for creedal probability could not codify logical truths about probabilities and could not coincide with logic of consistency and logic of truth....de Finetti and Savage rightly agreed with Ramsey that creedal probability judgments a lack truth-value so that a probability logic would be a logic of consistency or of coherence as Savage called it" (2010, 101).

[6] Essentially Keynes felt that events to which probabilities could be assigned formed a partially ordered set such that probabilities on events connected in the ordering could be compared, but probabilities on unconnected events could not. This was one of the reasons for distinguishing probability and something else; confidence, uncertainty, imprecision etc. Keynes illustrates this situation in the famous diagram in which different series of probabilities and their mutual relation are pictured (TP 1921, 39).

[7] Let M and N be lattices. A map $f : M \to N$ is said to be a *homomorphism* if and only if f is join-preserving and meet-preserving, that is for all x, y in M

$f/(x \sqcup y) = f(x) \sqcup f(y)$ and $f(x \sqcap y) = f(x) \sqcap f(y)$

A *bijection homomorphism* is a *lattice isomorphism.*

in what the keynesian theory does not conform to Ramseyian one. Following the seminal paper of Koopman, instead of assuming "a number (usually between 0 and 1) corresponding to the degree of a rational belief or credibility of the eventuality in question" (1940, 269), this aim is obtained by identifying the algebraic-axiomatic properties that support the relation between belief and probability in Ramsey's and Keynes's theories, that is the axiomatic structure that allows to pass from "the primal intuition of probability to that branch of the theory of measure that passes under the name of probability" (1940, 270).[8]

The paper shows that there is a particular class of abstract algebras able to represent Keynes's problem and shared properties with traditional Ramsey's probability theory. Such a class of abstract algebras is called *lattice*, where a lattice is a finite set ($\mathcal{X}$) of elements with an partial order ($\leq$) and an infimum and a supremum for each pair x and y of elements in the set $\mathcal{X}$.[9]

If the controversy between Keynes and Ramsey is reinterpreted by the lattice theory, in particular by bounded distributive lattices called *'pseudo distributive lattices'*, it is possible to represent Keynes's isomorphism given some conditions, as a weakened form of the Ramseyian one.

The paper is organized as follows. Section 2 describes the historical context of the debate on the foundation of probability and some crucial aspects in *TP*. Section 3 contains the formal definition of a lattice, then the representation of the Ramseyan set of beliefs by a *Boolean lattice* and Keynesian representation of belief by a *pseudo Boolean lattice*. Section 4 concludes.

2 Mathematics at the time of A Treatise on Probability

2.1 A Treatise on Probability

This paper is not about the philosophical debate on the origin of probability notion: creedal probability judgement, logical or epistemological probability, personal or subjective probability. It does not investigate if there is a logical interpretation of the formal calculus of probability or a problem of formal consistency and coherence, only. There are many authors that have been debating about that, involving: logicism, necessatarianism, pragmatism, positivism, operationism and so on. Debate is far from conclusive and '*pontoneers'* are disregarded, even if the notion of *confirmational commitment* and the distinction between *mandate probability judgments* and *permitted probability judgments* by probability logic with derived set of all logically permissible probabilities, introduced by Levi (2010) could be re-considered. Keynes separates the epistemo-

[8]In note 2, Koopman (1940) set out: "The phlosophical controversy here involved is very old and is still unsettled. Without attempting to give general references to the literature on this point we may cite the Introduction and First Chapter of J.M. Keynes' a Treatise on Probability (London, 1921)".

[9]A lattice is a partly ordered set X any two of whose elements x and y have a greatest lower bound or *meet* $x \cap y$ and a least upper bound. or *join* $x \cup y$.

logical foundation of his theory of probability (Part I) from its mathematical foundation by axioms and theorems (Part II). This paper is on the Part II, about Fundamental Theorems, only.

Keynes sets his dependence from the Russell's Principia Mathematica and, what is more relevant for critics, he recognizes the debt to W. E. Johnson, a *Cambridge Apostle* and professor of Keynes, Ramsey, Wittgenstein and Broad at the King's College (Sidgwick Lectureship in 1902)[10], who receives the final exposition of *TP* from his own hands. Keynes submitted a preliminary version of his book to Johnson, and said that it was "done for his criticism, and received the benefits not only of criticism but his own constructive exercises. The result is that in its final form it is difficult to indicate the exact extent of my indebtedness to him" (*TP,* 116).

Keynes gives axioms, definitions and theorems to make operative his notions and he says that theorems on combination are "based on a technical symbolic device known as the *Cumulative Formula* which is the work of Johnson" (*TP,* 122).

In *Chapter X-XII*, Keynes describes the preliminary properties of probability: certainty, impossibility, inconsistency and equivalence. Then he introduces addition and multiplication among probabilities. Addition is associative and commutative, but also distributive and by multiplication he derives independence and irrelevance. Crucially, Keynes declares that all these chapters are discussed and approved by Johnson, moreover from notes is evident that he is well aware of the current development of mathematical theory.

In *Chapter XII*, he describes the set of propositions or groups that are not self-contradictory and formally inconsistent with themselves, an order relation that ranks elements against one another, that is if an object is before it is less than another, two operations of meet and join and a complementary operation (*contradiction of an object a*). That is Keynes defines a set with a binary relation, two binary operations (addition and multiplication) and one unary operation (complementation). Keynes also describes certainty and impossibility as 1 and 0 probability, then he simply defines a bounded distributive lattice.

In *Chapter XIII,* Keynes establishes the fundamental theorems, supported by new axioms, of Certain or Necessary Inference, because "these theorems include those which the traditional logic has termed the Law of Thought, as for example the Law of Contradiction and the Law of the Excluded Middle" (*TP,* 121).

In *Chapter XIV,* Keynes characterizes the Law of Contradiction by setting, given a (conclusion) and contradictory $\bar{a}$ (Chapter X, 120 note 1), $(a+\bar{a})/h = 1$, thus "it is certain that either a or its contradiction $\bar{a}$ is true. This is the Law of Excluded Middle' (143). Then he sets the Addition Theorems such that '$(a+b)/h = a/h + b/h - ab/h$, which reduces to $(a+b)/h = a/h + b/h$, when a and b are mutually exclusive; and if $p_1 p_2 ... p_n$, form relative to h, a set of exclusive and exhaustive alternatives, $a/h = \sum_1^n p_r a/h$. Then the Theorem of

[10] Johnson published books on logic (1921-1924) and probability (1932) where he introduced the notion of exchangeability that was made famous by the de Finetti's representation theorem (1937).

Multiplication, Theorem of Inverse Probability, that represent the formal definition of the conditional probability when 'given the effect we investigate the cause" (*TP,* 149) and Theorem of Combination of Premises that combines hypothesis h_1 and h_2 in giving a conclusion. Results of the last theorem, says Keynes can be appreciated only by introducing a rule of reduction, that is the Cumulative Rule, that is the Johnson's Cumulative Formula, unpublished, even if "I had his permission to print below" (*TP* 150), Keynes said. In the Appendix to *Chapter XIV* - On Symbolic Treatment of Probabilities, Keynes discusses about the formal treatment of probabilities and cites Schroeder[11], who intended to publish a symbolic treatment of probability, but "whether he had prepared any manuscript on the subject before his death I do not know" (*TP,* 157). With axioms, theorems and "definitions combined with those of Chapter XII, it is easy to show (certainty being represented by unity and impossibility by zero) that we can manipulate according to the ordinary laws of arithmetic the 'numbers' which by means of a special convention we have thus introduced to represent probabilities" (*TP,* 159). Nevertheless, Keynes is interested in uncertainty or non-measurable probabilities or inexact numerical probabilities and says that "many probabilities, which are incapable of numerical measurement, can be placed nevertheless between numerical limits. And by taking particular non-numerical probabilities as standards a great number of comparisons or approximate measurements become possible" (*TP,* 160).

In *Chapter XV*, Keynes develops a method of approximation to non-numerical probability based on the Boole's Calculus.[12] This new method determines an interval of numerical probabilities by solving a system of equations (1921, 161).

Introduction Point 7 is about Chapter XV. Keynes states that he: "bring the non-numerical theory of probability developed in the preceding chapters into connection with the usual numerical conception of it, and demonstrate how and in what class of cases a meaning can be given to a numerical measure of a relation of probability. This leads on to what may be termed numerical approximation that is to say, the relating of probabilities, which are not themselves numerical, to probabilities which are numerical, by means of *greater and less*, by which in some cases numerical limits may be ascribed to probabilities which are not capable of numerical measures" (*TP,* 122).[13]

TP received many reviews, mostly from very famous philosophers (Russell),

[11] Schroeder died in 1902, but he gives some relevant contributions in algebra and what is more interesting for this paper, in lattice theory.

[12] In The Law of Thought (1854), G. Boole introduces minor limit and major limit of an event probability when the probability of an indefinite event is not elicited, but can be included into the interval [0,1]. In such a case "between these limits, it is certain that the probability sought must lie independently of all new experience which does not absolutely contradict the past" (1854, 268); and if the probability of a an event "cannot be numerically determined, we find, on assigning to it the limiting values 0 and 1, the following limits of probability of y [the given event]" (1854, 277). Boole continues in chapter XVIII by describing how to determine and use intervals of probability when new information is not determined. Hailperin (1986) states a possible solution to the Boole's Challenge Problem,

[13] Basili and Zappia (2009a, 2009b, 2010) and Brady and Arthmar (2012) show that non-numerical does not mean non-measurable in Keynes; on the contrary that it implies approximation by intervals, closed intervals (*TP,* 162 *note* 1).

statisticians, mathematicians and economists (Edgeworth and Ramsey). If the *TP* was such a relevant book, what made it invisible to science? Aldrich (2008) and Scheneider (2021) gave some reasons of *TP* disappearance: initial reviews ranged from dismissive to extremely positive; extensive logical notation, instability and multi definitions of symbols; no worked examples and not surprisingly the tone adopted by Keynes.[14]

Concerning the mathematical treatment of Keynes's probability notion, only a few reviewers commented on mathematical issues in *TP*, and among them Borel and Edgeworth are relevant because even if they put in evidence mathematical shortage, at the same time they know its scientific relevance. Borel (1924) "no mathematical contribution, [but] worth the necessary effort to understand an original work rich in novel insights" and Edgeworth (1922) "but in criticizing the mathematical portions of Mr. Keynes treatise it must be remembered that his object is not so much to sharpen the tools of the statistician as to show how they are used". Finally, there is the Ramsey's comments that focused on problems with Keynes's theory of probability.

2.2 The Foundational Crisis and the Reform of Mathematics

The crucial question is about the possibility of including an interval probability measure in Keynes's theoretical framework.[15]

Keynes published *TP* in 1921, Ramsey wrote the papers included in the *Foundation of Mathematics* between 1923 and 1929. In that period there was neither the axiomatic theory of lattice nor the mathematical representation of a probability as a positive normed measure, which came later. The controversy between Keynes and Ramsey involves the foundation of algebra of logic and measure theory but, unfortunately, formal theory is incomplete, sparing or absent, at that time.

Here it is a short reconstruction of the theoretical mathematical scenario where the Keynes-Ramsey's dispute takes place.

In 1847, De Morgan (1806-1871) and Boole (1815-1864) published *Formal Logic* and *The Mathematical Analysis of Logic*, respectively. Both of them were interested in going beyond the limits of traditional syllogistic logic and developed "mechanical modes of making transictions, with a notation which represents our head work" (De Morgan's letter to Boole, 28 November 1847, 22 17).[16] Boole introduced the operations of addition. multiplication and difference moreover,

[14]Four reviewers (Fisher, Bowley, Wilson and Pearl) commented on the tone adopted by Keynes, often using quotes from the Treatise. Wilson, for example observed 'Never perhaps since ancient biblical times has such a redoubtable army of philistines been so deftly slain (Judges, XV, 15)' after quoting Keynes statement (180) that it may 'be safely said that the principal conclusions on the subject set out by Condorcet, Laplace, Poisson, Cournot and Boole are demonstrably false' (Schneider 2021, 956).

[15]There is a strict analogy with the quantum mechanics where the probability of an observable is not atomic, that is concentrated in a single point, but lies in a region of the set of states.

[16]Boole had a lifelong correspondence with De Morgan, beginning in 1842.

by the symbolizing of particular propositions, he originateda symbolic algebra, which was the germ of the consequent Boolean Algebra.

De Morgan's early researches are on mathematics, not on mathematical logic. He was awarded a medal by the Royal Society for an article on differential equations and wrote two books on differential equations before obtaining a position in mathematics at the Queen's College of Cork (Ireland), in 1849. In Cork, De Morgan issued *An Investigation of the Laws of Thought,* his fundamental book on Logic, in 1854. De Morgan's idea is to represent logic using a notion of algebra, ordinary language is considered inadequate for investigation of logic and in the preface of *Laws of Thought* he wrote "the design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of a Calculus, and upon this foundation to establish the science of Logic and construct its method". De Morgan's *S4* (1860) can be considered the landmark of the logic of quantified relations that allows proving of De Morgan's Laws and other more sophisticated theorems, such as the *Theorem K.*

The book of Misak (2020), the papers of Batemann (2021) and Gerrards (2023) provide the opportunity for a new insight the dispute that has been dividing economists and historians since the 1920s. Ramsey explicitly refers to Peirce (1923, 92) when divides arguments into two classes "analytic, explicative, or deductive and amplificative, synthetic, or (loosely speaking) inductive" (1926, 186). Finally, in *The Logic of Truth*, Ramsey says "what follows to the end of the section is almost entirely based on the writings of C.S. Peirce" (1926,194 *Note 2*).

Peirce calls his theory of logic Pragmatism.[17] In *What Pragmatism is,* Peirce states "now quite the most striking feature of the new theory was its recognition of an inseparable connection between rational cognition and rational purpose; and that consideration it was which determined the preference for the name pragmatism " (*The Monist* 1905, 163). Nevertheless, influence of the Peircean formal logic on the Ramseyean theory of probability has been mis-regarded, for a long.

This paper only considers the Peirce's contributions to the calculus of binary relatives and quantified propositional logic. His famous seminal theorem on complemented algebras (1880) is examined in the perspective of a development of the Boolean algebra, that is the theoretical framework of Ramsey.

J. Venn, S. Jevons, a fellow of De Morgan, and Peirce develop the ideas of Boole and De Morgan about the calculus of relations, quantified relations and symbolic algebras. In particular, Peirce combines linear algebra, calculus of propositions and relatives operations. In 1870, Peirce publishes *Logic of Relatives* where he introduces the notion of the relative product, esistential quantifiers and properties of relations such as symmetry, transitivity, reflexivity etc. In *The Algebra of Logic* (1885) Peirce gives a definition of finite set that rested on the De Morgan's 'bicopular syllogism' but do not depend on a

[17] Dewey set that "the term and the idea were suggested to him by a reading of Kant, the idea by the *Critique of Pure Reason* and the term by the *Critique of Practical Reason* (1916, 710).

natural numerical sequence.[18] In *Algebra of Logic: A Contribution to the Philosophy of Notation* (1885), Peirce proposes developing an algebra adequate to the treatment of all problems of deductive logic. According to ordinary logic, he assumes that "a proposition is either true or false, and no further distinction is recognized" (Peirce 1865, 183). Peirce sets out that uniquely complemented algebras are Boolean algebras. Even if his proof is wrong, this idea persists in the lattice theory up to the counterexample due to Dilworth (1945) that shows the necessary condition of distributivity for a Boolean algebra.

In *Vorlesungen über die Algebra der Logik* (1890-1895), Schroeder, who everywhere declares his indebtedness to Peirce, extends Peirce's theory of relatives and gives an axiomatic foundation of the lattice-theoretic representation of Boolean algebra that can be defined as the proto-Boolean algebra of logic.[19] In 1898, Schroder proves the equivalence between the Peircean demonstration of a finite set and the Dedekind's one.

In the same period, a number of mathematicians, such as Pasc and Hilbert, were liberating geometry from the Greek space-intuition, by which greek philosophers and mathematicians conceived the relation of their abstract geometry and ordinary space, and introducing axiomatic theory based on axioms and propositions (Heyting 1930).

In *Sets of Postulates for the Algebra of Logic* (1904), Huntington offers three sets of postulates for defining an algebra of logic, in the sense of Peirce and Schroeder. Finally, in 1913, Sheffer introduces the term Boolean Algebra and offers his axiomatization (*Sheffer stroke*).[20]

Borel and Lebesgue write about measure theory, integration and function theory after *TP* was written and published, so its reviews. In 1928, in Germany, von Mises published *Probability, Statistics and Truth,* his manifesto about the concept of probability and its scope, where the probability is defined as the limiting value of an observable relative frequency for a particular attribute in a mass (collective) phenomenon. As a matter of fact, probability is about relations existing between physical quantities in: games of chance, certain mechanical and physical phenomena and certain problems relating to social mass phenomena.[21]

The alternative approach is due to Kolmogorov, but his fundamental original German monograph *Grundbegriffe der Wahrscheinlichkeitrechnung* appears in the Ergebnisse Der Mathematik (*Foundations of the Theory of Probability*) only in 1933 and the Russian translation in 1936.[22] Alternatively with respect

[18] In 1888 Dedekind defined an infinite set as a set that can be put into one-to-one correspondence with part of itself

[19] In the *Volume I* of his *Algebra der Logic*, Schroeder presented an axiomatization of Boolean algebra based on classes and propositions using two operations $(+)$ and $(\cdot)$ In *Lecture* VI, he discussed the question raised by the Peirce's Theorem on unique complementation.

[20] Sheffer (1913) "A set of five independent postulates for Boolean algebras, with application to logical constants"

[21] 'Our probability theory has nothing to do with questions such as:is there a probability of Germany being at some time in the future involved in a war with Liberia?' (von Mises 1928, 9).

[22] In 1925, Andrei Kolmogorov published the first (partial) formalization of intuitionistic logic, and his technical result anticipated Gödel-Gentzen's 'double negation translations' for

to von Mises, Kolmogorov's scope is to give an axiomatic foundation for the frequentist theory of probability. Kolmogorov states that the theory of probability has to be defined by a system of sets that satisfy certain conditions, that is the elements to be studied, their basic relation and a handful of axioms by which those relations are governed. Kolmogorov thinks that the natural place where the basic concepts of probability theory is among the general notion of modern mathematics, but such a task "would have been a rather hopeless one before that introduction of Lebesgue's theories of measure and integration". In the preface of *Foundations of the Theory of Probability*, he states that after Lebesgue's publication of his investigations, the analogies between measure of set and probability of an event, and between integral of a function and mathematical expectation of a random variable, became apparent. However, he says, it is necessary "to make the theory of measure and integration independent of geometric elements which were in the foreground with Lebesgue. This has been done by Frechet". Indeed, Frechet was the first to introduce the notion of metric space (or Hausdorf space) and the topological concepts of compactness, completeness, and separability, in his doctoral thesis (1906), and published his seminal book *Les Espaces Abstracts Topologiquement Affines*, in 1926.

In 1933, Birkhoff introduces the term lattice to describe an independent mathematical theory. In 1936 in two papers about an alternative space model for quantum mechanics and new geometrical structures, von Neumann gave a definition of lattice. In 1936, in the context of the Eintein-Pocholsky-Rosen and Bohr's controversy about the quantum mechanical description of physical reality, Birkhoff and von Neumann published *The Logic of Quantum Mechanics* and developed a lattice theory that was not distributive but ortho-complemented, only.[23] Finally, Birkhoff shows some lattice applications (1938) and publishes the first monography on lattice theory, in 1940.

Summarizing, lack of a formal definition of a consistent interval probability measure is a problem related to Keynes's knowledge of mathematics or is the limit of contemporary mathematical theory?

Keynes was not able to be precise and formal in his definition of probability under uncertainty represented by closed intervals of probabilities, because such a mathematics had not been discovered, simply.

3 Algebras and Lattices

This Section follows the Birkhoff-von Neumann approach and a lattice is defined. It is shown that all the formal structure of the Ramseyan theory and Keynesian theory is common but they are different with respect to complementation, that in the case of lattice isomorphic with a Boolean algebra of sets (events) corresponds to passage to the set complement.

arithmetic (On the principle of the excluded middle).

[23] Th distributive law was substituted by the De Morgan's Laws and complementation by orthomodular complementation. As a consequence the probability theory for quantum mechanics was made non-additive.

In a nutshell bounded lattice is a partially ordered set that admits both a top and a bottom element, and each pair of judgments has both a least upper bound (or join) and a greatest lower bound (or meet).[24] Observe that any totally ordered set such as, say, a bounded interval of numbers, is an instance of a bounded lattice.

3.1 Ramsey's Representation of Belief

A Boolean Algebra is the mathematical structure where the properties of the Ramseyan notion of probability: addition, multiplication, complementation and additivity are defined.[25] Additivity and unique complementation imply the Law of Excluded Middle and indeed, in *Mathematical Logic* (1926), Ramsey states that "Brouwer, the leader of what is called the intuitionist school, whose chief doctrine is the denial of the Law of Excluded Middle, that every proposition is either true or false....Brouwer would refuse to agree that either it was raining or it was not raining, unless he had looked to se" (1926, 65).[26] Brouwer, one of the greatest mathematicians of the 20th century introduces modern topology and the fixed-point theorem, is the founder of intuitionistic mathematics that underlies intuitionistic logic. In 1898, Brouwer defined his criticism of Law of the Excluded Middle by the so called Weak Counterexamples (The Unreliability of the Logical Principles).[27]

Let $\mathcal{X} = (X, \leqslant, \sqcup, \sqcap, u, z)$ be a bounded distributive lattice and $\mathcal{X}$ be an algebraic structure. Assume that $u = 1$ (unit element) and $z = 0$ (null element), then the algebraic structure $\mathcal{X}$ can be formulated in terms of some operations: the sure event or true (u), the null event or false (z), then it is bounded (constant or nullary operation), a complementation for each element X (unary operation), two maps that combine objects in the set corresponding to union or disjunction

[24] Let $\mathcal{X} = (X, \leqslant)$ be a partially ordered set (poset) of belief, $\mathcal{X}$ is called a bounded distributive lattice, namely a set X endowed with a partial order $\leqslant$ (i.e. a reflexive, transitive and antisymmetric binary relation) if and only if: (i) X includes both a maximum u and a minimum z; (ii) for any $x, y \in X$ both lattice join $\sqcup$, denoted as $x \sqcup y$, such that $x \leq x \sqcup y$ and $y \leq x \sqcup y$ and lattice meet $\sqcap$, denoted as $x \sqcap y$, such that $x \sqcap y \leq x$ and $x \sqcap y \leq y$ are well-defined binary operations on X, (iii) for all v in X, $z \leq v$ and $v \leq u$ and (iv) the join and meet operation satisfy the distributive identities namely $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$ and $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$, for every $x, y, z \in X$.

[25] The first studied lattice was the complemented distributive lattice considered by Boole, and called Boolean lattice in his honor.

[26] Ramsey was so sure that the Law of Excluded Middle was true, that in absence of a proof, he appealed to Aristotelean argument "if a proposition is neither true or false, let us call it doubtful; but if the Law of Excluded Middle be false, it need not be either doubtful or not doubtful, so we shall have not merely three possibilities, but four, that it is true, that it is false, that it is doubtful, and that it is neither true, false nor doubtful. And so on *ad infinitum*" (1926, 66). The Law of Excluded Middle and derived additivity are axioms and theorems that supported the Ramsey's theory of choice where "any definite degree of belief implies a certain measure of consistency namely willingness to bet on a given proposition at the same odds for any stake....such a consistency between odds acceptable on different proposition as shall prevent a book being made against you—[option] is based throughout on the idea of mathematical expectation" (1926, 183).

[27] Brouwer introduced his conception of the relations between mathematics, language, and logic, in 1907.

($\sqcup$) and intersection or conjunction ($\sqcap$) or binary operation. In the case of interest to this problem, in the bounded distributive lattice have to be defined various, at least two, concepts of complementation or negation.[28]

Let $\mathcal{X} = (X, \leqslant, \sqcup, \sqcap, z, u)$ be a bounded distributive lattice and x be an element of X. Then y in X is said to be a complement of x if and only if $x \sqcup y = z$ and $x \sqcap y = u$, i.e. the set complementation holds. $\mathcal{X}$ is said to b complemented if and only if each of its elements is complemented. A complemented distributive lattice is said a boolean lattice and it is well known that each boolean lattice is uniquely complemented.[29]

$\mathcal{X} = (X, \leqslant, \sqcup, \sqcap, -, u, z)$ is a Boolean lattice and the unary operation of complementation or negation is represented by $(-)$.

If $u = 0$ and $z = 1$, the boolean lattice becomes $\mathcal{X} = (X, \leqslant, \sqcup, \sqcap, -, 0, 1)$, it is straightforward show that Ramsey's framework satisfies the Law of the Excluded Middle, in fact the $x \sqcup -x = 1$, that is the Sure Event and additivity holds, always.

Stone's Theorem (1936)[30] establishes an isomorphism between Boolean lattices, a special kind of distributive lattice, and Boolean algebras of events. Boolean algebra of events is the events space of classical probability theory and is the algebraic structure in which Ramsey introduces his axioms to characterize a probability measure.[31]

3.2 Keynes's Representation of Belief

Keynes assumes a set of imprecise and approximate belief that do not support boolean complementation. Nevertheless, it is possible to show that Keynesian set of belief is a pseudo Boolean algebra of events and introduce the notion of *weak-complementation*.

Consider an algebra $\mathcal{A}$, sometimes a σ-algebra, of subsets of X, denoted as events, with the relation of inclusion (order relation) and the standard operations of intersection and union, then $\mathcal{A}$ constitutes a bounded distributive lattice.

Let $H \subseteq X$ be an event, then $H^W = H^C \backslash H^I$ is called a weak-complement of H, where H^I is the set of uncertain opportunities or indecisive eventualities of H relative to the assumed weak complementation. Then for each event H,

[28] A nullary operation is a constant. A unary operation on the set X is a bijective function $f : X \to X$. Complementation maps an element x in X to its negative (*negation* x). A binary operation in X is a function $f : X \rtimes X \to X$ such that for each x and y in X it follows that $F(x, y)$ is defined in X

[29] In a Boolean lattice every element a has a unique complement, and it is also relatively complemented, that is it has a relative complement in any interval containing it.

If $x \in \mathcal{X}$ and x' is the complement such that $x \cap x' = 0$ and $x \cup x' = 1$, for $a \leq x \leq b$, then $(a \cup x') \cap b = a \cup (x' \cap b)$ is a relative complement of x.in the interval $[a, b]$.

[30] Stone's representation Theorem for Boolean lattice, (1936) *Each Boolean lattice is isomorphic, to a Boolean algebra of events.*

[31] Koopman (1940) defines the numerical probability and its properties in a boolean algebra that is isomorphic to a boolean lattice in virtue of the Theorem of Representation of Boolean Algebras of Stone.

there exists a partition of X given by $X = H^C \cup H^W \cup H^I$, where H^I is not necessarily the empty set.

The *weak-complement* is indicated by $(\neg_{\mathcal{W}})$ and the bounded distributive pseudo boolean lattice $\mathcal{X} = \langle X, \leq, \sqcup, \sqcap, \neg_{\mathcal{W}}, z, u \rangle$ is called *weak-complement* bounded distributive lattice. Let $H \subseteq X$ be an event, then H^W is called a weak-complement of H, if and only if it has the following properties: $H^{\mathcal{W}} \subseteq H^C$, for every event $H \subseteq X$, where H^C is the usual theoric-set complement, $\neg_{\mathcal{W}}(\neg_{\mathcal{W}} H) = H$ (*regular complement*) and for any pair of events $H, K \in X$ with $H \subseteq K$, let $H_I = H^C \backslash H^{\mathcal{W}}$ and $K_I = K^C \backslash K^{\mathcal{W}}$ be the *sets uncertain opportunities* or *indecisive eventualities,* then $H_I \subseteq K_I$ (*monotone*).

By interpreting eventualities as the causal factors leading to the occurrence of a given random phenomenon, it is introduced the set of indecisive eventualities (uncertain opportunities) associated with an event H.[32]

In such an algebra, a class of interval probability measures can be established, as Keynes suggests to represent uncertainty in *Chapter XV*, then the interval probability P_{H_I} defined on $\mathcal{A}$ by

$$P_{H_I}(H) = \Big[P(H), P(H) + P(H_I) \Big] \tag{1}$$

is the interval probability measure according to P and relative to (the uncertainty determined by) H_I. The interval $P_{H_I}(H)$ is said to be the measure (according to P) of H relative to H_I.

As a result, the width $|P_{H_I}(H)|$ of the interval $P_{H_I}(H)$ is the probability according to P of the event H_I and the interval measure coincides with the classical one according to P if and only if the set of uncertain opportunities of H (i.e., the uncertainty induced by the causes determining H and realizing the random phenomenon) is P-negligible. As a consequence, it is possible to interpret the keynesian uncertainty by a class of interval probabilities measures, defined when a model of uncertainty is determined by either a partition of X, that allows to represent a weak complementation (Basili and Pratelli 2024).

Birkhoff's Theorem (1937)[33] assures that there is an isomorphism between a finite distributive lattice, non-necessarily complemented, and a distributive algebra of partially ordered sets. Roughly speaking, the Birkhoff's theorem assures that a finite distributive lattice and a finite partial order are just the same thing.[34]

[32] Basili and Pratelli 2024.

[33] An element a of a lattice $\mathcal{X}$ is called join-irreducible if $x \cup y = a$ implies $x = a$ or $y = a$. If all chains in a lattice $\mathcal{X}$ are finite, then every $a \in \mathcal{X}$ can be represented as a $a = x_1 \cup x_2 \cup ... \cup x_n$.of a finite number of join-irreducible elements.

(Birckoff's Representation Theorem 1937): *Any finite distributive lattice is isomorphic to the lattice of lower sets of the partial order of its join-irreducible elements.*

[34] The elements of every finite distributive lattice can be represented as the lower sets of an underlying partial order that may be uniquely determined from the lattice (it's the lattice's ordering on the subset of elements that can't be formed as joins of smaller elements). The join operation on these lower sets is just set union, and the meet operation on these lower sets is just set intersection. Conversely the lower sets of any partial order, with these operations, form a distributive lattice

The new notion of imprecise probability represented by interval probability measures allows disentagling the Keynesian example of barometer and black clouds: "is our expectation of rain, when we start out for a walk, always more likely than not, or less likely than not, or as likely as not? I am prepared to argue that on same occasions none of these alternatives hold, and that it will be an arbitrary mater to decide for or against the umbrella. If the barometer is high, but the clouds are black, it is not always rational that one should prevail over the other in our mind, or even that we should balance them" (Keynes 1921, 28).

Such an example shows that the negation of a given cause does not necessarily imply the non-occurrence of the represented event.[35] In fact assuming the partition $Z = \{Z_0, Z_1\}$ with $Z_0 = \{w_{0,1}, w_{1,0}\}$, $Z_1 = \{w_{0,0}, w_{1,1}\}$ to describe the weak complementation and, consequently, uncertain events. In particular, the Law of the Excluded Middle does not hold because the (weak) complementary event of $\{w_{1,0}\}$ is $\{w_{0,1}\}$ and Z_1 represents the set of indecisive eventualities of $\{w_{1,0}\}$. As a fact, when facing discordant situations like $w_{0,1}$ and $w_{1,0}$, it is not possible to infer anything about whether these are due to a distorted or mis-revealed occurrence of agreeing causes in Z_1. It the same way, when a decision-maker faces $w_{1,1}$ and $w_{0,0}$, which are agreeing situations, it does not allow them to deduce whether these are mis-reliable occurrences of discordant situations ($w_{1,0}$ and $w_{0,1}$) or not (Basili and Pratelli 2024).

Even if set complementation does not fully define the set of states of the world, as observed in Ramsey, to establish a formal representation of uncertainty and subsequently evaluate it through imprecise probability, it is necessary to introduce a weak complement, even in a rough form. This allows the formulation of a model of uncertainty. The novel concept of imprecise probability leads to a clear conclusion: since the notion of weak (relative) complementation is not unique, implying various models to describe uncertainty, a large family of interval probability measures exists that can model and assess sets of eventualities for a random phenomenon. Consequently, the framework proposed by Ramsey is merely a specific case of Keynes's framework, as the Law of the Excluded Middle is derived by assuming the set complementation (Basili and Pratelli 2024).

4 Concluding Remarks

The controversy between Keynes and Ramsey occurred in 1920s. This paper shows that introducing weak complementation in a Boolean algebra it is possible to represent Keynesian theory of probability by a bounded distributive lattice. As a consequence, bounded and distributive lattices allow one to induce a formal relationship between the set of beliefs and the algebra of events, where the notion

[35] In the framework of imprecise probabilities or interval probability measures, the problem of the umbrella is a situation where there are four relevant causes for raining:
1) barometer low and no black clouds in the sky, denoted as eventuality $w_{0,1}$;
2) barometer high, many black clouds in the sky, denoted as eventuality $w_{1,0}$;
3) barometer low, many black clouds in the sky, denoted as eventuality $w_{0,0}$;
4) barometer high, no black clouds in the sky, denoted as eventuality $w_{1,1}$.

of probability is defined. In Ramseyan Boolean lattice the probability is precise, unique and additive. In the Keynesian pseudo Boolean lattice uncertain beliefs are represented by interval probability measures. Remarkably, in such a pseudo Boolean lattice, a unique interval probability measure can be derived when a probability measure and the related weak complementation (or the model of uncertainty) are predetermined.

In summary, the isomorphism between belief and probability in Keynes's framework cannot be demanded when the notion of weak (relative) complementation is not predefined. However, once assumed, there exists a unique interval probability distribution that represents imprecise belief, that is an isomorphism or the Arabian Phoenix sought by Ramsey. Introducing a lattice representation, the century-old controversy on the theoretical representation of probability can be reconsidered by concluding that Ramsey's theory is only a special case of Keynes's theory.

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